International Journal of Mining and Geo-Engineering

IJMGE

IJMGE 53-1 (2019) 91-97

DOI: 10.22059/ijmge.2019.242221.594699

Modeling and prediction of time-series of monthly copper prices

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Article History:

Received: 22 September 2017,

Accepted: 13 February 2019.

ABSTRACT

One of the main tasks to analyze and design a mining system is predicting the behavior exhibited by prices in the future. In this paper, the applications of different prediction methods are evaluated in econometrics and financial management fields, such as ARIMA, TGARCH, and stochastic differential equations, for the time-series of monthly copper prices. Moreover, the performance of these methods is investigated in predicting the time-series of monthly prices of copper during early 1987 till late 2014. This study shows that the mean of about thousand runs using the Stochastic Differential Equations (SDE) method for 33 out of range cases gives better forecasting results for copper price time-series in comparison to traditional linear or non-linear functional forms (such as ARIMA and TGARCH) to model the price movement.

Keywords : *Copper, Price forecasting, ARIMA, TGARCH, Stochastic differential equations*

1. Introduction

Although prediction is a key element in taking management decisions, it is always erroneous. The most significant researches undertaken in the field of future price forecasting are those corresponding to stock return or prices in stock exchange markets with major part of mathematical financial activities focused on this subject.

Metals are important traded commodities, and thus, forecasting their price has important commercial and also economic implications. Despite the significant number of studies on metals and their price, the precise pricing mechanism in metals market has not been presented [1]. Dooley & Lenihan [2] argued that metals price tends to be the major factor causing variability in revenues from mining operations. Price forecasting is important to investigate whether a deposit can be exploited economically.

System recognition principles and dynamic model governing stock price and return prediction undertaken via linear, non-linear, or random modelling approaches, can be extended to other pricing sectors such as future price prediction for metals such as iron, copper, gold, and even in the oil and gas industry. Metals prices are the result of complex market dynamics and stochastic economic processes, which makes the price forecasting process difficult [3]. Labys used a structural time-series model to forecast the monthly prices of copper, lead, tin, zinc, and other primary commodities. Dooley & Lenihan [2] presented 3-month forward and 15-month forward prices for lead and zinc. The results of comparing and evaluating the ARIMA and lagged forward modeling approaches show that, in mining industry, price forecasting is extremely difficult and these techniques are not capable of forecasting the price in mining systems. Kriechbaumer et. al [4] provided the motivation for combining wavelets with the ARIMA models to forecast the monthly base metal prices. Accordingly, the normal ARIMA models have shown to be rather unsuitable for predicting the monthly based metal prices.

In traditional time-series-related predictions, future values are assumed to follow linear or non-linear trends followed by previous values [2]. Different tests have been already developed to evaluate and characterize the nature, linearity, non-linearity, chaos, and randomness of the time-series.

There are several reports that have used non-linear models of the ARCH family for the economical time-series modeling [5-9]. Tan et. al. [10] used combination of ARIMA and generalized autoregressive conditional heteroskedasticity (GARCH) models for electricity price forecasting. Zhang & Tan [11] forecast Day-ahead electricity price using EGARCH model. So far, the use of autoregressive conditional non-linear ARCH family models in metal price time-series forecasting is not common and has not been reported.

Long-term forecasts are more unreliable than short-term ones and it should be remembered that no forecasting methodology will be fully accurate all of the time so there are risks associated with using them. As Van Rensburg & Bambrick [12] pointed out, forecasting remains an art rather than a science. Future values are assumed to follow linear or non-linear trends processed by common models such as auto regressive integrated moving average (ARIMA), generalized autoregressive conditional heteroskedasticity (GARCH), etc. For these models to be efficient, there should be a linear relationship (except for ARCH model family) along with datasets of normal distribution and sustainability – these are normally not meet in financial markets [13-15]. Stochastic differential equations are recommended for more complex cases where the exhibited non-linear behavior looks completely of a stochastic nature [16-18].

In this paper, the important role of copper in the industry as well as the excessive need for modeling and forecasting its price are taken into account in drawing a future image of the design of mining systems. Therefore, in this study, different forecasting methods are discussed among which the SDE method is suggested as an appropriate forecasting method.

2. Time-series analysis of Box – Jenkins model (linear models)

The time-series analyses were theoretically and practically developed during 1970s to perform the prediction and control tasks. Some analyses

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are commonly related to data that lack the independency and are sequentially dependent on one another. This interdependency along sequential observations is used for in prediction tasks [15]. The linear time-series models include the auto regressive (AR) model, the moving average (MA) model, the auto regressive moving average model (ARMA), and the autoregressive integrated moving average (ARIMA). In these models, the variable under consideration is expressed in terms of its historical values along with previous interference or error terms. The Box – Jenkins approach determines how the time-series under consideration does or does not follow any of the mentioned linear models [13].

3. Nonlinear models

Time-series models are generally analyzed based on the variance homology assumption which may not be valid for many time-series datasets including economic ones. Therefore, the rational method is to use models where in heteroskedasticity constraints are considered when having models fitted. One of well-known families of such non-linear models is called autoregressive conditional heteroskedasticity (ARCH) including the symmetric GARCH model. In this model, the conditional variance (i.e. instantaneous or short-term variance) is assumed to be a function of stopping the conditional variance and prediction error values, with the variance of errors for each period being a function of its preceding values. In this way, one can undertake dynamic predictions in time-series models based on their average and variance values. In the symmetric GARCH model, in contrary to asymmetric models and systems, an identical variance variability is assumed for either of positive or negative shocks. Later on, the TARCH model was proposed to model the effects of good and bad events on the fluctuations. This model is characterized by the conditionality of standard deviation rather than variance [9]. In total, so far, different models of ARCH family have been proposed to model non-linear series such as, TARCH, TGARCH, EARCH, GARCH-M and etc.

4. Stochastic differential equations method

The dynamic nature of time-series of prices often follows stochastic and chaos behaviors. Considering random fluctuations in stock prices, using stochastic differential equations can be an efficient workaround for the modeling and prediction of the economic time-series. These models were first introduced to the economy literature by the works of Black & Scholes [19], and Merton [20, 21] which were dedicated to the modeling of stock prices in terms of a geometrical motion stochastic differential equations. In this model, a return on stock or stock price are assumed with the stock prices following a log-normal distribution of fixed fluctuations [22]. Even with non-fixed fluctuations, one can use the average values without any interference with the job flow. Accordingly, if the price fluctuations follow a given time-dependent function, the model recovery principles will remain unchanged with the instantaneous fluctuations considered in terms of the average instantaneous fluctuations [19]. Principally, the future price prediction is highly sensitive to the initial price conditions, so that a given level of error in the present period may generate very significant levels of error in the periods to come.

In spite of its chaos or random nature, stochastic differential equations method provides suitable applicability in terms of predicting future economic parameters. The stochastic trend of the time-series is introduced into the model in terms of random Weiner process, and due to this trend, the model results are unstable to overcome which problem or simulations are launched for many runs [22]. In simulations, rather than introducing a mathematical relation to solve the problem, the model is analyzed and tested under stochastic conditions for many times, so as to achieve reliable results on actual performance of the system [23]. The corresponding Black – Sholes – Merton stochastic differential equations to model the stock prices is as follows:

$$dS(t) = \alpha S(t)dt + \sigma S(t)dw(t), \quad S(0) = S_0 \tag{1}$$

where S(t) denotes the stock price at time t, α represents the mathematical expectation of return on stock, σ is the instantaneous return on stock, and w(t) refers to the geometric Brownian motion or Weiner process which reproduces the fluctuating behavior of the time-series S(t) [18, 23]. One of apparent limitations of the model is the normality assumption of return on securities prices accounted for as a required condition for the model to be applicable. However, further studies have shown the efficient applicability of this model to non-normal distributions as well [19]. Interested readers on the Brownian motion and the Weiner process are referred to references [7, 16, 24] for more details.

With p(t) denoting price levels, α and σ being the price drift and price volatility at time t, respectively, Black – Sholes – Merton idea implies that p(t) will be satisfied in the Brownian motion equations, so that the stochastic differential equations can be solved to determine the stochastic process which represents the behavior of price variable throughout the time as follows:

$$dp(t) = \alpha p(t)dt + \sigma p(t)dw(t), \qquad p(0) = p_0 \tag{2}$$

According to appendix A, for $p(t_0) = p_0$, $t_0 = 0$, we have:

$$p(t) = p(0)e^{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma w(t)} \tag{3}$$

Eq. (3) demonstrates the dynamic nature of price behavior throughout the time. There are different methods to calculate α and σ, among which the simplest one is to use their preceding values according to Eq. (4) and (5) [22].

$$\hat{\alpha} = \frac{\sum_{i=0}^{N-1}\left[p(t_{i+1}) - p(t_i)\right]}{\sum_{i=0}^{N-1} p(t_i)} \tag{4}$$

$$\hat{\sigma} = \sqrt{\frac{\sum_{i=0}^{N-1}\left[p(t_{i+1}) - p(t_i)\right]^2}{\sum_{i=0}^{N-1} p(t_i)^2}} \tag{5}$$

5. Modeling of time-series of monthly copper prices

Fig. 1 shows the chart of time-series of monthly copper prices from early 1987 till late 2014 [25]. Based on the existing experiences and comparing the figure with other featured series, the time-series of copper prices is evidently stochastic and irregular. In order to model the time-series of copper prices, some of the most important common above-mentioned models were used; the time-series was extrapolated using these models.

5.1. Prediction of time-series using linear and non-linear models

The EViews software package was utilized to model the time-series using some linear models of Box – Jenkins family (such as ARMA and ARIMA) as well as some non-linear models (such as ARCH, GARH, TARCH, TGARCH, etc.).

The following steps were undertaken to perform the modeling task using the mentioned software:

The determination of overall series trend using the linear model.

The use of Dicky – Fuller test and other tests to evaluate whether the series is stationary or not.

Making the series stationary by taking different logarithms, if required.

The determination of autocorrelation function (ACF) and partial autocorrelation functions (PCF) to obtain the optimum stops for either of ARIMA models.

The determination of the series variance using the heteroskedasticity variance test for the complementary recognition, once finished with determining the linear model.

Undertaking the required steps to eliminate the heteroskedasticity variance effect, once it was confirmed further proving the non-linear condition.

In the absence of symmetry across the series under study, the TARCH and TGARCH models can also be used. Other non-linear models of the ARCH family are also recommended under special conditions and on-demand. The following presents a descriptive summary of the modeling results of copper prices time-series using the EViews software.

Initially, through investigating the series diagram and the results of the Dicky- Fuller test, the data was once changed to achieve a stationary time-series; the results of the Dicky- Fuller test confirmed the series to become stationary. Then, by introducing different autoregressive terms along with an adequate moving average and an ordinary least squares (OLS) estimation, and based on the Akaike and Shwarz criteria while considering the fitness evaluation measures of Dublin – Watson, standard deviation of error, and correlation coefficient, the ARIMA method was selected as the proper model. After selecting the ARIMA model as the proper model via Box – Jenkins method, the Lagrange multiplier heteroscedasticity test was used to investigate the data on monthly prices of copper. The null hypothesis was defined as the hypothesis testing the non-existence of ARCH within the data, as the rejection of this hypothesis proves the ARCH effect and further necessitating its removal. The Lagrange multiplier heteroscedasticity test showed that the series suffer from a non-linear structure, so that, due to the variance fluctuations, linear models may not be able to adequately address the problem; this necessitates the use of non-linear models. To separate the non-linear effects from the series of monthly copper prices, the predictions were conducted using the ARCH, GARCH, TARCH, and TGARCH models. Knowing that the conditional variance is not constant in the ARCH non-linear model family, the maximum likelihood method is used to predict the models. Generally speaking, the ARCH non-linear model family has an equation for the average, and another one for the variance. The corresponding equations of the average model the copper price changes, while that of the variance models the copper price fluctuations. Based on the experiences gained in linear estimation methods, the decreasing trends observed in Akaike and Shwarz indicate the improving nature of the model; while in non-linear estimation methods (maximum likelihood), an increasing trend may provide the same indication. Table 1 reports a summary of the results of modeling using different methods; these are explained in further details in appendix B.

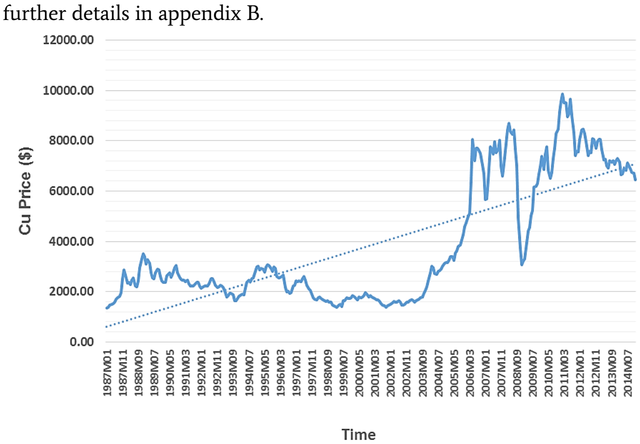

Fig. 1. Chart of time- series of monthly copper prices from early 1987 till late 2014.

5.2. Model prediction using stochastic differential equations

Knowing that the common linear and non-linear models used in the analysis of time-series cannot predict the non-linear structure within the data, more complex models such as stochastic differential equations are required. In order to model copper prices, first, parameters α and σ are set to $\hat{\alpha} = 0.0039$ and $\hat{\sigma} = 0.070$, respectively, based on the time-series of monthly prices (including 336 data points).

Table 1. A summary of the results of prediction models for time-series of copper prices (exported from EViews software).

Model	First order	Durbin – Watson measure	Akaike criterion	Shwarz criterion	Multiplier significance level	Comments
ARIMA	ARIMA(2,1,3)	1.97	14.25	14.32	Significant	Although the model is stationary, it suffers from the variance heteroskedasticity. In particular, as the time passes and the prices increase or change, the long-term model estimation results are highly erroneous
TGARCH	TGARCH(1,1)	1.66	14.05	14.16	Significant	In spite of being stationary, as well as the elimination of conditional variance heteroskedasticity of the model, its accuracy is on doubt due to instability of unconditional long-term variance of the time- series

By substituting these values together with the initial copper price (5830) into the corresponding stochastic differential equations, the following equation is developed:

$$p(t) = 5830\exp\left[\left(0.0039 - \tfrac{1}{2}0.070^2\right)t + 0.070w\ (t)\right] \qquad (6)$$

Assuming a normal distribution of the Weiner multiplier, 1000 runs of the simulation were undertaken. Fig. 2 shows the simulation results in the form of the Brownian motion for 72 months. Further indicated values in this figure are the average simulated values. The average simulation results indicate the time-series follow and increase a trend throughout the time.

Fig. 3 indicates the estimation and forecast results of the monthly copper prices corresponding to an interval outside the range of study, i.e. from 2015 till 2020. In addition, the annual World Bank forecast is presented for comparison.

The results of different prediction method for 2015, 2016, and the first 9 months of 2017 are presented in Table 2 (33 cases). The best results are the average values estimated by stochastic differential equations. Compared to non-linear TGARCH, linear ARIMA provided higher accuracy for the recent 33 months. However, taking the heteroskedasticity variance of the ARIMA model into account, its accuracy is somehow in doubt. In addition, the non-linear TGARCH model was confirmed to provide lower accuracy than the other models.

Based on the results presented in Fig. 3 and Table 2, as well as the undertaken qualitative and quantitative evaluations, the following summery can be concluded:

- ARIMA-based models also suffer from inefficiencies stemming from failing to account for intense changes in the variance of time-series of copper prices. Failing to account for either of short-term or long-term intense fluctuations will contribute to the tendency towards predicting around a fixed value or impose a linear increasing trend in long-run.
- Long-term series variance is known as the unconditional variance, while the variable short-term variance is called the conditional variance. It may be the case that a series has its long-term variance fixed yet, its short-term one changes in which the time-series enjoys the conditional heteroskedasticity variance. Changes in the unconditional variance and its increasing trend throughout the time along with the conditional variance changes may result in unconventional estimates of commonly near zero values.

Regarding the smaller changes in early years while larger changes in the latter ones, some models of the ARCH family including the ARCH, GARCH, and TARCH models fail to model the series of monthly copper prices; while the response by TGARCH was somehow adequate to the recent case. In general, application of non-linear ARCH methods and their derivatives as heteroskedasticity variance models do not respond in cases similar to time-series of copper prices where the unconditional long-term variance exhibits a large deal of fluctuations.

- The stochastic nature of copper series, intense fluctuations, and the changing variance may justify the performance of stochastic differential equations method. Comparing the estimation results of the recent 33 months further highlighted the superiority of this method over its counterparts.

Due to the stochastic nature of the series, intense fluctuations throughout the time, and lack of future information, a comprehensive quantitative measure for the evaluation of the performance of well-known models is yet to be proposed. As such, it seems necessary to account for the stochastic nature of the time-series of copper prices and implementation of experts' opinions when one is to draw a proper scenario for the future of copper price.

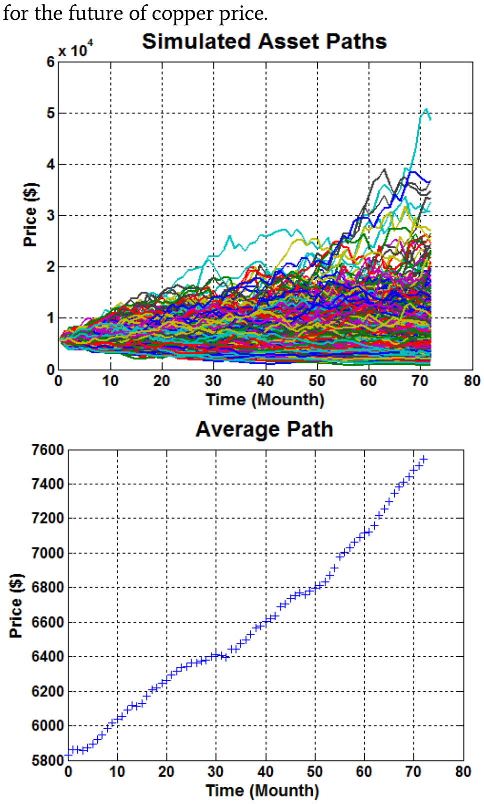

Fig. 2. Simulation results of the SDE monthly copper prices over 72 months.

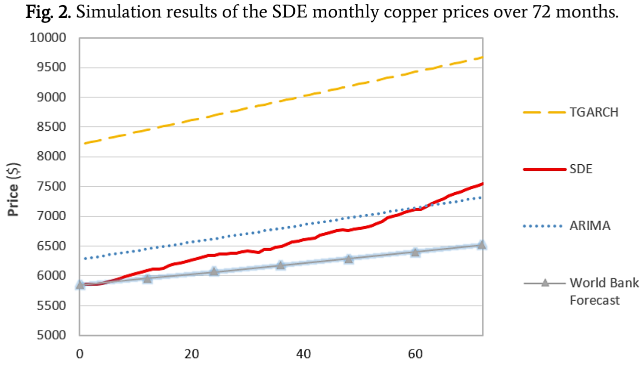

Fig. 3. Prediction results of monthly copper prices corresponding to an interval outside that of available data, i.e. from 2015 till 2020.

Table 2. Different prediction method results for 2015, 2016, and the first 9 months of 2017 (33 cases).

Time	Price ($/t)	ARIMA	TGARCH	SDE
Jan. 2015	5830.54	6305.053	8227.286	5862.613
Feb. 2015	5729.27	6319.529	8247.582	5862.485
March 2015	5939.67	6334.005	8267.894	5858.464
April. 2015	6042.09	6348.48	8288.19	5870.119
May 2015	6294.78	6362.956	8308.502	5892.549
June 2015	5833.01	6377.432	8328.798	5919.627
July 2015	5456.75	6391.908	8349.109	5948.184
Aug. 2015	5127.3	6406.384	8369.406	5984.842
Sept. 2015	5217.25	6420.86	8389.717	6014.606
Oct. 2015	5216.09	6435.336	8410.014	6039.24
Nov. 2015	4799.9	6449.812	8430.325	6055.978
Dec. 2015	4638.83	6464.288	8450.622	6091.784
Jan 2016	4471.79	6478.764	8470.933	6117.629
Feb 2016	4598.62	6493.24	8491.23	6110.946
Mar 2016	4953.8	6507.716	8511.54	6129.225
Apr 2016	4872.74	6522.192	8531.838	6170.893
May 2016	4694.54	6536.668	8552.148	6206.115
Jun 2016	4641.97	6551.144	8572.446	6218.982
Jul 2016	4864.9	6565.62	8592.756	6243.332
Aug 2016	4751.67	6580.096	8613.054	6263.154
Sep 2016	4722.2	6594.572	8633.364	6291.735
Oct 2016	4731.26	6609.048	8653.662	6313.25
Nov 2016	5450.93	6623.524	8673.971	6336.7
Dec 2016	5660.35	6638	8694.271	6341.769
Jan 2017	5754.56	6652.475	8714.579	6362.587
Feb 2017	5940.91	6666.951	8734.879	6361.367
Mar 2017	5824.63	6681.427	8755.187	6373.589
Apr 2017	5683.9	6695.903	8775.487	6382.083
May 2017	5599.56	6710.379	8795.795	6400.39
Jun 2017	5719.76	6724.855	8816.095	6412.17
Jul 2017	5699.48	6739.331	8836.403	6406.762
Aug 2017	5978.6	6753.807	8856.703	6395.388
Sep 2017	6478.35	6768.283	8877.01	6443.896
Mean Absolute Error (MAE)		1112.86	2894.28	831.021
Mean Square Error (MSE)		1670178	10445064	986932.2
Mean Absolute Percentage Error (%) (MAPE)		20.90	54.36	15.61

6. Conclusions

Although the inefficiency of linear models such as ARIMA and the non-linear approaches such as TGARCH is evident, but as a general recommendation, it is suggested to use the evaluation tests on the normality or non-normality, the chaos or non-chaos, and the stochastic or deterministic nature of the time-series under study to select a final method to undertake the analysis and modeling tasks. Choosing a well-suited modeling approach to the series nature will prepare a basis to achieve more convenient and more accurate predictions.

It is highly difficult to present a definite conclusion regarding which long- term forecasting produces are acceptable of the predictions. However, according to the results of the current research, considering the mean value of around one thousand runs using the SDE method for 33 cases of out of range, the forecasts using the SDE method provide a superior result to the ARIMA and TGARCH prices. The highest forecast accuracies were achieved using SDE in addition to the lowest MAPE with about 15.60 %. MAPE of ARIMA and TIGARCH were calculated 20.90 % and 54.36 %, respectively. Employing the SDE model, the highest predictive power was achieved for both MAE and MSE. The predictive power of the TGARCH model was drastically low and inacceptable. Therefore, in the absence of comprehensive method for forecasting, SDE model is a good choice as a promising technique for copper price forecasting.

In the stochastic differential equation method, two terms are usually used to express the pricing process; one of these two terms correspond to the average instantaneous price changes, while the other one represents the instantaneous fluctuations in the pricing process. In more complex cases, it is also possible to introduce the mutation and shock terms to the related equations. These configurations may justify the use of stochastic differential equations method to predict the time-series of monthly copper price that has an absolutely stochastic nature.

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Appendix A.

The Ito lemma is probably the most important stochastic arithmetic theorem, which is used to solve stochastic differential equations. Using Ito lemma along with variable exchange in Eq. (A.1), differencing rule of stochastic differential equations, the following relations can be written to find the solution of Eq. (2) [26]:

$$Y\left(t\right)=U\left(p\left(t\right)\right)=\ln p\left(t\right) \qquad \text{(A.1)}$$

$$dY\left(t\right)=\frac{\partial U}{\partial t}\left(t,p\left(t\right)\right)dt+\frac{\partial U}{\partial p\left(t\right)}\left(t,p\left(t\right)\right)dp\left(t\right)+\frac{1}{2}\frac{\partial^2}{\partial p\left(t\right)^2}\left(t,p\left(t\right)\right)dp\left(t\right)^2$$

(A.2)

$$dp(t)^2 = \left(\alpha p(t)dt + \sigma p(t)dw(t)\right)^2$$
$$= \alpha^2 p(t)^2 dt^2 + 2\alpha p(t)\sigma p(t)dtdw(t) + \alpha^2 p(t)^2 dw(t)^2$$
(A.3)

Based on mathematical relationships related to stochastic differential equations, we have:

$$\prec dt, dt \succ = \prec dw(t), dt \geq 0; \prec dw(t), dw(t) \geq dw(t)^2 = dt$$
(A.4)

As such, Eq. (A.3) can be incorporated into Eq. (A.5) to achieve the following relationship:

$$dp(t)^2 = 0 + 0 + \sigma^2 p(t)^2 dw(t)^2 = \sigma^2 p(t)^2 dt$$
(A.5)

Similarly, it can be expressed that:

$$dY(t) = d\left(\ln p(t)\right) = 0 + \frac{dp(t)}{p(t)} - \frac{1}{2}\frac{1}{p(t)^2}\sigma^2 p(t)^2 dt = \frac{dp(t)}{p(t)} - \frac{1}{2}\sigma^2 dt$$
(A.6)

$$\frac{dp(t)}{p(t)} = \alpha dt + \sigma dw(t)$$
(A.7)

$$d\left(\ln p(t)\right) = \alpha dt + \sigma dw(t) - \frac{1}{2}\sigma^2 dt = \left(\alpha - \tfrac{1}{2}\sigma^2\right)dt + \sigma dw(t)$$
(A.8)

Integrating both sides within [0, t] we will have:

$$\int_0^t d\ \ln\left(p(t)\right) = \int_0^t \left(\alpha - \tfrac{1}{2}\sigma^2\right) + \int_0^t \sigma dw \qquad \text{(A.9)}$$

Therefore, for $p(t_0) = p_0, t_0 = 0$, we have:

$$p(t) = p(0)e^{\left(\alpha - \frac{1}{2}\sigma^2\right)t + \sigma w(t)} \qquad \text{(A.10)}$$

Appendix B.

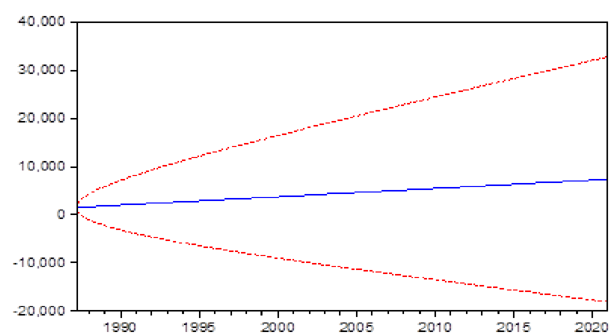
B.1. Summary of ARIMA model

Dependent Variable: D(CU)
Method: Least Squares
Date: 11/13/15 Time: 17:02
Sample (adjusted): 1987M04 2014M12
Included observations: 333 after adjustments
Convergence achieved after 53 iterations
MA Backcast: 1987M01 1987M03

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	14.47595	23.31188	0.620969	0.5351
AR(1)	-1.545208	0.202385	-7.634974	0.0000
AR(2)	-0.662969	0.178376	-3.716692	0.0002
MA(1)	1.930142	0.198606	9.718444	0.0000
MA(2)	1.299463	0.241516	5.380436	0.0000
MA(3)	0.334832	0.069430	4.822569	0.0000

R-squared	0.154326	Mean dependent var	14.95715
Adjusted R-squared	0.141395	S.D. dependent var	322.6050
S.E. of regression	298.9287	Akaike info criterion	14.25614
Sum squared resid	29220195	Schwarz criterion	14.32476
Log likelihood	-2367.648	Hannan-Quinn criter.	14.28350
F-statistic	11.93477	Durbin-Watson stat	1.977198
Prob(F-statistic)	0.000000		

Inverted AR Roots	-.77+.26i	-.77-.26i	
Inverted MA Roots	-.52+.33i	-.52-.33i	-.90



Forecast: CUFARIMA
Actual: CU
Forecast sample: 1987M01 2020M12
Adjusted sample: 1987M04 2020M12
Included observations: 333

Root Mean Squared Error	1689.515
Mean Absolute Error	1391.271
Mean Abs. Percent Error	49.71790
Theil Inequality Coefficient	0.194072
Bias Proportion	0.000471
Variance Proportion	0.409581
Covariance Proportion	0.589948

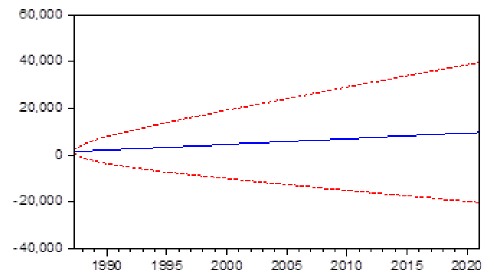
B.2. Summary of TGARCH model

Dependent Variable: D(CU)
Method: ML - ARCH (Marquardt) - Normal distribution
Date: 11/13/15 Time: 17:04
Sample (adjusted): 1987M04 2014M12
Included observations: 333 after adjustments
Failure to improve Likelihood after 9 iterations
MA Backcast: 1987M01 1987M03
Presample variance: backcast (parameter = 0.7)
GARCH = C(7) + C(8)*RESID(-1)^2 + C(9)*RESID(-1)^2*(RESID(-1)<0) +
C(10)*GARCH(-1)

Variable	Coefficient	Std. Error	z-Statistic	Prob.
C	20.30395	28.49847	0.712457	0.4762
AR(1)	-0.921465	0.092499	-9.961843	0.0000
AR(2)	0.051613	0.102725	0.502445	0.6154
MA(1)	1.094657	0.050846	21.52902	0.0000
MA(2)	0.241893	0.031618	7.650372	0.0000
MA(3)	0.135776	0.038554	3.521705	0.0004
Variance Equation				
C	67093.47	11704.49	5.732283	0.0000
RESID(-1)^2	0.541496	0.223083	2.427326	0.0152
RESID(-1)^2*(RESID(-1)<0)	-0.139187	0.223727	-0.622130	0.5339
GARCH(-1)	0.007017	0.068539	0.102382	0.9185

R-squared	0.090781	Mean dependent var	14.95715
Adjusted R-squared	0.076878	S.D. dependent var	322.6050
S.E. of regression	309.9563	Akaike info criterion	14.05204
Sum squared resid	31415844	Schwarz criterion	14.16640
Log likelihood	-2329.665	Hannan-Quinn criter.	14.09764
Durbin-Watson stat	1.662164		

Inverted AR Roots	.05	-.97	
Inverted MA Roots	-.05+.37i	-.05-.37i	-.99



Forecast: CUFTARCH	
Actual: CU	
Forecast sample: 1987M01 2020M12	
Adjusted sample: 1987M04 2020M12	
Included observations: 333	
Root Mean Squared Error	1898.641
Mean Absolute Error	1528.587
Mean Abs. Percent Error	66.52750
Theil Inequality Coefficient	0.193898
Bias Proportion	0.269486
Variance Proportion	0.075300
Covariance Proportion	0.655214