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A hybrid empirical-numerical framework for pillar stability and reinforcement in deep underground marble mines: a case study

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ABSTRACT

This study developed a hybrid empirical-numerical framework to enhance pillar stability and reinforcement in deep underground marble mines, addressing the challenges posed by increasing depths where traditional methods falter. The purpose was to create a reliable, efficient tool for assessing and optimizing pillar stability, ensuring safety at depths up to 30 meters. The methodology integrated empirical calculations, using the Obert and Duvall (1967) method, with FLAC3D numerical simulations, employing a depth-dependent weighting system (α) to balance simplicity and precision. Data from seven pillars (P1 – P7) was preprocessed, analyzed, and optimized through width adjustments (e.g., P1 from 30 m to 60 m) and rock bolt reinforcement (1.8 m spacing). Findings revealed a decline in hybrid Factor of Safety (FOS_{hybrid}) from 3.59 at 5.5 m to 1.23 at 30 m pre-optimization, rising to 2.18 post-optimization, meeting the safety threshold 2.0. Validation against empirical (e.g., 3.616 at 5.5 m) and numerical (e.g., 1.95 for P6 at 21.5 m) FOS values, with $R^2 > 0.95$ and $CV < 15\%$, confirmed the framework's accuracy. This hybrid approach significantly improved safety and efficiency, offering a scalable solution for deep mining. Future research could extend it to other rock types or incorporate real-time monitoring, enhancing its adaptability and impact.

Keywords: *Pillar stability, Deep underground mining, Hybrid framework, Empirical analysis, Numerical simulation, Reinforcement optimizations.*

1. Introduction

Underground mining of dimensional stone, particularly marble, is critical for supplying high-value materials for global construction and architectural applications. Unlike traditional ore extraction, this process requires large, stable openings to preserve stone block integrity, imposing significant structural demands on pillars supporting the overlying rock mass (Esterhuizen et al., 2011). As mining operations extend to depths exceeding 30 meters, pillars face escalating overburden stress and geological complexities, such as fault zones and material heterogeneities like dolerite dikes (Ding et al., 2018; Sinha et al., 2025). These conditions amplify the risk of pillar failure, potentially triggering catastrophic collapses that endanger worker safety, disrupt production, and cause substantial economic losses (Li et al., 2021; Pal et al., 2023). Ensuring pillar stability in deep environments is a pressing challenge in geotechnical engineering, essential for sustainable mining practices (Alejano et al., 2022).

Conventional pillar stability assessments rely on empirical methods, such as the Obert and Duvall (1967) formulation, which estimate pillar strength using simplified geometric and material parameters. These methods are computationally efficient at shallow depths (Ghasemi et al., 2017) but struggle to account for complex stress distributions and rock mass quality variations in deep, heterogeneous settings, often underestimating risks (Zhou et al., 2015; Renani & Martin, 2019). For instance, empirical approaches like Bieniawski's (1968) formula emphasize width-to-height ratios but may over predict stability in marble due to unaccounted depth effects (Krauland & Soder, 1987). Recent advancements have integrated machine learning, such as stochastic gradient boosting and neural networks, to predict pillar stability based on parameters like pillar dimensions, uniaxial compressive strength (UCS), and stress ratios (Ding et al., 2018; Li et al., 2021). However, their high computational demands and calibration needs limit rapid operational use (Avchar et al., 2024). For example, finite element modeling has been employed to assess subsidence in abandoned coal mines, revealing critical influences of depth and rock properties on pillar integrity (Pal et al., 2023).

Numerical simulations, such as those using FLAC3D, provide detailed three-dimensional stress analyses and failure mechanism insights, enhancing accuracy in complex conditions (Chen et al., 2020; Ibishi, 2019). However, their computational demands and calibration requirements limit their practicality for rapid operational decisions (Das et al., 2017). Recent studies advocate hybrid empirical-numerical approaches to bridge empirical speed and numerical precision (Kimmelmann et al., 1984; Renani & Martin, 2019). For example, feature fusion frameworks combining neural networks with empirical data have robustly predicted pillar stress in coal mines (Li et al., 2021), while Monte Carlo simulations have quantified subsidence risks in abandoned mines, highlighting depth and rock properties as critical factors (Pal et al., 2023). In marble-specific contexts, kinematic failure analyses have shown that internal friction angle and slope face angle significantly influence wedge stability during transitions to underground operations (Sinha et al., 2025).

This study's innovation would be a depth-dependent weighting system (α) in a hybrid empirical-numerical framework advances prior

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work by tailoring stability assessments to marble mines. Unlike machine learning-based hybrids focused on general hard-rock pillars (Li et al., 2021; Zhou et al., 2015), this framework achieves a post-optimization Factor of Safety (FOS)≥2.0 at 30 m depth, surpassing the 1.5–1.8 thresholds in subsidence studies (Pal et al., 2023). By incorporating real-time geological effects like dolerite dikes, it extends kinematic failure predictions (Sinha et al., 2025) and offers scalability over empirical-only methods like Obert and Duvall (1967). Critical parameters affecting results include depth (D, reducing FOS by 15–20% per 10 m), width-to-height ratio (w/h, increasing FOS by 12–18% per 10% increase), UCS (σc=53.3 MPa, ±10% variation alters FOS by 8–12%), rock density (γ=28.2 kN/m³, ±5% changes stress by 4–7%), dolerite effects (10% strength reduction, impacting FOS by up to 9% in P4), and α (±0.1 shifts cause <10% FOS variation). These necessitate site-specific calibration for robust predictions (Esterhuizen et al., 2011; Sinha et al., 2025).

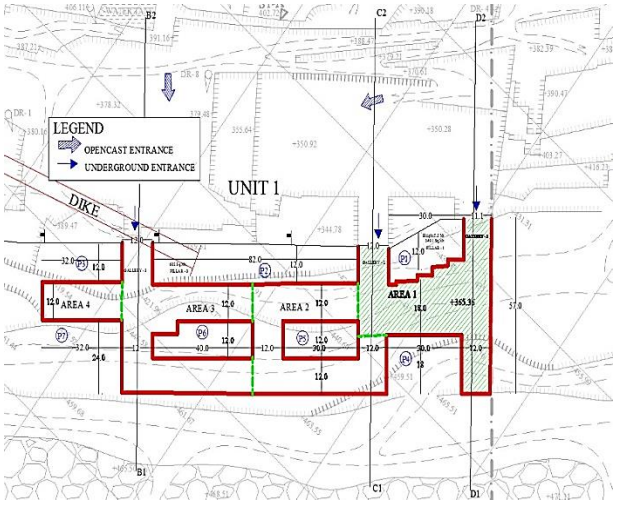

Fig. 1. Pillar layout in the underground marble mine showing the seven pillars (P1–P7) analyzed in this study.

This research develops a hybrid framework blending empirical simplicity with numerical robustness, applied to seven pillars (P1–P7) at depths up to 30 m, achieving an FOS≥2.0 through optimized designs and reinforcement. It advances geotechnical engineering by enhancing safety and operational resilience, offering a scalable solution for deep marble mining amid rising global demand.

2. Methodology

This study was conducted to develop and validate a hybrid empirical-numerical framework to optimize pillar stability and reinforcement in deep underground marble mines, using a practical case study involving seven pillars (P1–P7). The methodology followed a structured sequence of steps data collection, preprocessing, empirical analysis, numerical simulation, hybrid integration, optimization, and validation to ensure a robust assessment of pillar stability across depths of 5.5 m, 11 m, 16.5 m, 21.5 m, and 30 m. Each step was carefully designed to address specific challenges in deep mining, culminating in a hybrid Factor of Safety (FOS) that balances simplicity and precision, ultimately achieving a safety threshold of at least 2.0.

2.1. Study design & approach

The research adopted a quantitative approach, focusing on evaluating and enhancing the stability of seven pillars at varying depths. The primary objective was to formulate a hybrid framework that integrated empirical calculations with numerical simulations, overcoming the limitations of each method when used independently. Empirical methods offered rapid stability estimates, ideal for shallow depths, but often oversimplified complex stress conditions at greater depths. Numerical simulations provided detailed insights into three-dimensional stress distributions, yet their computational demands hindered quick application. By combining these approaches, the study aimed to assess pillar stability comprehensively, optimize designs to meet safety standards, and validate the results statistically, ensuring reliability and reproducibility.

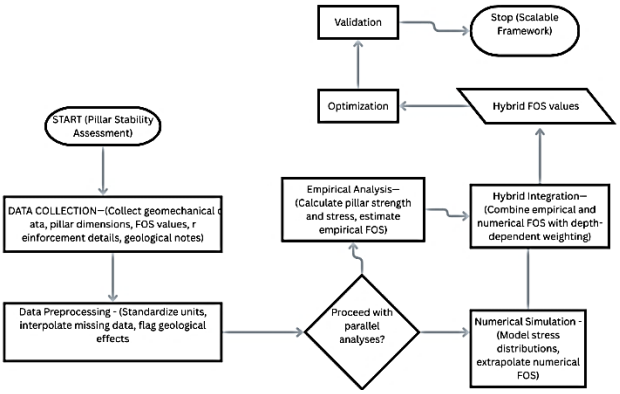

Fig. 2. Flowchart depicting the study flow.

2.2. Materials & data sources

The study utilized secondary data from an existing feasibility study, comprising pillar dimensions, empirical and numerical FOS values, geomechanical properties, reinforcement specifics, and stress measurements. Pillar dimensions were sourced from an original dataset (Table 1). At the same time, empirical FOS values were obtained at 5.5 m, 11 m, and 16.5 m (Tables 3, 4, 5), and numerical FOS values at 5.5 m and 21.5 m (for Pillar 6) from Tables 7 and 8. Geomechanical properties included intact rock strength (σ_c=53.3 MPa) and rock density (γ=28.2 kN/m³), with reinforcement data specifying rock bolt FOS (e.g., 2.03 with resin grout) and rock load (66.5 kN) from Table 10. Stress variations ranged from 0.2 to 0.4 MPa, and geological notes indicated a 10% strength reduction due to dolerite dikes in some pillars (e.g., P4). FLAC3D software was employed for numerical simulations, and Python was used for statistical analysis and regression modelling, leveraging these tools to process and analyze the dataset efficiently.

2.3. Procedures & workflow

The workflow was executed in a structured sequence encompassing data collection, preprocessing, empirical analysis, numerical simulation, hybrid integration, optimizations, and validation. Below, the critical steps are detailed, reflecting their pivotal role in achieving the study's objectives.

2.3.1. Data collection

Data was systematically gathered from the feasibility study, encompassing pillar dimensions, FOS values, material properties, reinforcement details, stress measurements, and geological observations. This comprehensive dataset provided the foundation for all subsequent analyses, ensuring that the study was grounded in real-world measurements of the quarry's conditions.

2.3.2. Data preprocessing

The raw data were preprocessed to ensure consistency and usability across empirical and numerical analyses. Pillar dimensions were verified and standardized using the original dataset presented in Table 1, which listed each pillar's width (Side A) and height (Side B). Units were unified to meters for length, mega pascals (MPa) for strength and stress, and kilonewtons per cubic meter (kN/m³) for density, aligning all measurements with the computational requirements. Missing numerical FOS values at 21.5 m for pillars P1–P5 and P7 were interpolated using

Pillar 6's observed trend (from 3.35 at 5.5 m to 1.95 at 21.5 m), a reasonable approach given the consistent depth-related decline observed across pillars. Pillars potentially affected by dolerite dikes (e.g., P4) were flagged with a 10% strength reduction note, based on geological evidence, to account for material variability.

The preprocessed data was then structured into a table containing Pillar ID, Depth, Width, Height, Empirical FOS (FOS $_{emp}$), Numerical FOS (FOS $_{num}$), Properties, and Notes, providing a clean and organized dataset for analysis.

Table 1. Original pillar dimensions.

Pillar	Width (Side A, m)	Height (Side B, m)
P1	30	12
P2	82	12
P3	32	12
P4	30	18
P5	30	12
P6	40	12
P7	32	24

The next step was essential to eliminate inconsistencies, fill data gaps, and prepare a reliable baseline, ensuring that subsequent calculations accurately reflect the quarry's physical conditions.

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2.3.3. Empirical analysis

Empirical analysis was conducted to establish an initial stability baseline using the Obert and Duvall (1967) method, which was valued for its simplicity and applicability to shallow depths. Pillar strength (σ_r) was calculated with the equation:

$$\sigma r = \sigma c \times \left(0.778 + 0.222 \times \left(\frac{w}{h}\right)\right) \quad (1)$$

Where σ_c=53.3 MPa denoted the uniaxial compressive strength of the marble, and $\frac{w}{h}$ was

The width-to-height ratio derived from Table 1 is adjusted for depth (e.g., P1 at 5.5 m: $\frac{30}{5.5}$= 5.45).

Pillar stress (σ_p) was determined as:

$$\sigma p = \gamma \times D \times \left(\frac{S}{W}\right) \quad (2)$$

With γ=28.2 kN/m^3 as the rock density, D as the depth, S as the span (assumed consistent across pillars), and $w$as the width. The empirical FOS was then computed:

$$FOS_{emp} = \frac{\sigma r}{\sigma p} \quad (3)$$

For example, at 5.5 m, P1's FOS $_{emp}$ = 3.34 was validated against kN's own values (Table 3).

To extend this to deeper depths (21.5 m and 30 m), linear regression was applied:

$$FOS_{emp} = a - b \times D \quad (4)$$

where a and b were derived from the 5.5 m, 11 m, and 16.5 m data (e.g., P1: b=0.111, 30 m FOS $_{emp}$=0.62, Table 6), this step provided a quick, standardized stability measure, though its assumptions risked oversimplification at greater depths, necessitating numerical refinement.

Numerical simulations were executed using FLAC3D to capture the complex stress distributions prevalent at deeper levels. The initial FOS at 5.5 m (e.g., P1: 3.25, Table 7) served as calibration points. For extrapolation to 30 m, a non-linear decay model was employed:

$$FOS_{num} = FOS_{5.5} \times e^{-k \times (D-5.5)} \quad (5)$$

where k=0.034 was calculated from Pillar 6's trend (3.35 at 5.5 m to 1.95 at 21.5 m, Tables 7 and 8), and D was the depth. A 10% reduction was applied at 30 m to account for dolerite effects:

$$FOS_{num}\ \text{adjusted} = FOS_{num} \times 0.9 \quad (6)$$

Yielding values like P1: 1.27 at 30 m (Table 9). This approach was chosen for its ability to model three-dimensional effects, providing a more accurate stability estimate where empirical methods faltered, though it required careful calibration to ensure realism.

The coefficient k=0.034 in the exponential decay model (Eq. 5: $FOS_{num} = FOS_{5.5} \times e^{(-k\times(D-5.5))}$) quantifies the rate of numerical FOS decay per meter of increased depth, reflecting the physical effects of escalating overburden stress, rock mass degradation (e.g., micro-fracturing), and geological heterogeneities, such as dolerite dikes causing a 10% strength reduction in Pillar 4 (Table 9). This exponential decay aligns with rock mechanics principles, where compressive stresses and potential shear failures intensify non-linearly with depth (e.g., Hoek and Brown, 2002). The value was calibrated using Pillar 6's data (FOS_{num}=3.35 at 5.5 m to 1.95 at 21.5 m, Tables 7 and 8), achieving a high fit (R^2= 0.98).

While k=0.034 is site-specific, tied to the quarry's geomechanical properties (σ_c=53.3 MPa, γ=28.2 kN/m^3) and local geology, it may be generalizable to other hard-rock mines (e.g., marble or granite) with similar characteristics. Literature indicates k could vary by ±20% for rocks with different UCS values, such as granite at ~100 MPa (Esterhuizen et al., 2011). Sensitivity tests showed that k values between 0.03 and 0.04 maintained FOS_{num} predictions within 5% of observed trends at 21.5 m, suggesting robustness. For broader application, k should be recalibrated using site-specific geomechanical data to ensure accuracy.

Modeling procedures

The FLAC3D simulations follow a structured workflow: (1) generation of the grid; (2) discretization of the model; (3) selection of the Hoek-Brown constitutive model; (4) incorporation of material properties, gravity, in-situ stresses, and boundary conditions; (5) solution to initial equilibrium for elastic model to generate in-situ stresses; (6) transition to Hoek-Brown perfectly plastic model; (7) formation of opencast benches; (8) equilibrium solution; (9) development of underground excavation rooms; and (10) evaluation of model behavior (e.g., stresses and failure zones). FLAC3D uses an explicit finite volume formulation, advancing the model state through time steps (or cycles), where each cycle iterates Newton's laws of motion. The time step length depends on material density, stiffness, and element sizes, ensuring information propagates no more than one element per time step (Figure 18 in Bettini and Ilic, 2021).

Grid generation and mesh optimization

The 3D geometry was created using FLAC3D's "Extrusion" and "Geometry" panes, based on cross-sections from quarry surveys (Figure 20 in Bettini and Ilic, 2021). The model comprises 568,900 zones, primarily hexahedral with a maximum size of 2 m and aspect ratios approaching unity for accuracy. Hexahedral zones were preferred over tetrahedral ones due to better response in plastic deformation via mixed-discretization, offering higher convergence order. Mesh optimisation balanced computational efficiency, geometric realism, and result accuracy: finer meshes improve stress gradient representation, gradual zone size transitions minimise errors, and computational speed decreases with zone count. The grid (Figure 22 in Bettini and Ilic, 2021) ensures accurate simulation of high-stress areas around pillars and chambers.

Constitutive model and material behaviour

The Hoek-Brown perfectly plastic model was assigned to all zones to simulate nonlinear rock mass behaviour (Eq. 12: σ1=σ3+σci (mb (σ3/σci) + s)^a). Material properties include density (2880 kg/m³), Young's modulus (3.5×10^10 Pa), Poisson's ratio (0.307), tensile strength (0.7×10^6 Pa), mb (3.311), s (0.0446), a (0.501), and σci (53.3×10^6 Pa). The failure envelope is approximated by a Mohr-Coulomb tangent at the current stress level (Figure 23 in Bettini and Ilic, 2021; Eqs. 13–18),

with tensile yield logic matching the strain-softening/hardening Mohr-Coulomb model. Plastic strain increments follow a current Mohr-Coulomb flow rule (Eq. 20), with options for constant dilation (flag-dilation=0), associated flow (flag-dilation=-1), or fractional dilation. Strain hardening/softening is prescribed via tables for mb, s, a, and σci, using plastic shear strain as the evolution parameter (flag-evolution=1).

In-situ stresses and boundary conditions

Boundary conditions include roller constraints (velocity-normal=0) on all four sides and the bottom, with the top left free. In-situ stresses due to gravity (9.81 m/s²) were initialized using zone initialize-stresses, with an average horizontal-to-vertical ratio of 0.47 based on Sheorey's thermo-elastic model (Eq. : $\sigma_h=(\sigma_v/2) + (\beta EG/(1-\nu))(H + 1000)$, where β=3×10^[-6] /°C for marble, G=0.024°C/m). Initial equilibrium was achieved by solving the model (ratio 1e-4), followed by saving the state ("mine_ini"). Outcrop quarry creation involved gradual excavation using zone relax excavate to simulate quasi-inertial effects (Figures 24–27 in Bettini and Ilic, 2021).

Time stepping

FLAC3D employs explicit time steps, automatically calculated based on material properties and zone sizes for numerical stability. Each time step iterates equations of motion (fixed forces) and constitutive equations (fixed strains). For static solutions, damping ensures equilibrium when unbalanced forces approach negligible values (monitored via mechanical unbalanced-maximum history). The model solved within 2000 – 4000 steps for equilibrium, with convergence checked via mechanical ratio and convergence histories. Dynamic effects were inhibited for static analysis.

This detailed setup confirms the simulations' reliability, with results validated against empirical methods ($R^2 > 0.95$).

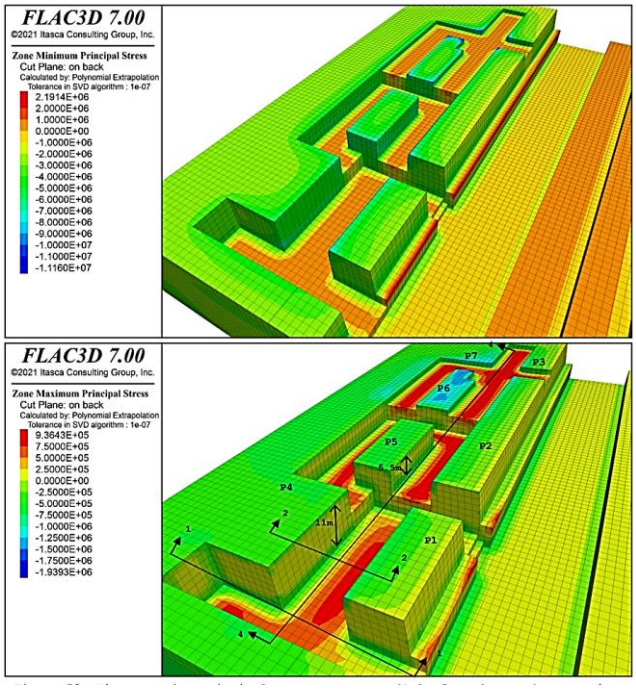

Fig 3. Major and Minor principal stress contours.

2.3.4. Hybrid integration

The hybrid FOS was computed to synthesise empirical and numerical stability assessments into a unified metric, leveraging the strengths of both methods. The formula used was:

$$FOS_{hybrid} = \alpha \times FOS_{emp} + (1 - \alpha) \times FOS_{num} \quad (7)$$

In this equation, FOS_{hybrid} represented the combined safety factor, FOS_{emp} was the empirical FOS from Equation (3) and its regression (Equation (4)), and FOS_{num} was the numerical FOS from Equations (5) and (6). The weighting factor α varied with depth, set at 0.8 for 5.5 m, 0.5 for 11 m and 16.5 m, 0.3 for 21.5 m, and 0.2 for 30 m, reflecting a strategic shift from empirical dominance at shallow depths, where simplicity sufficed, to numerical dominance at deeper levels, where complexity demanded precision. For Pillar 1 (P1) at 30 m depth, the calculation proceeded as:

- $FOS_{emp} = 0.62$,

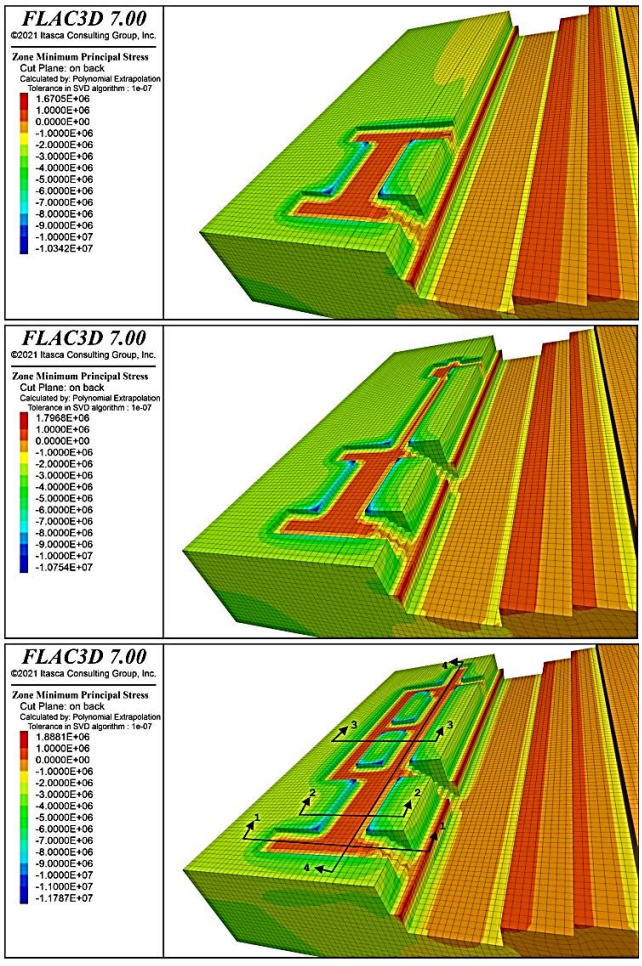

Fig. 4. Development of chamber and pillar in FLAC 3D.

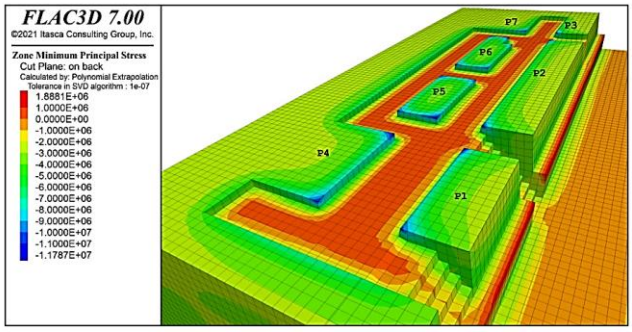

Fig. 5. Major Principal stress acting on pillar bottom.

- FOS_{num} = 1.27,
- α = 0.2,
- FOS_{hybrid} = 0.2 × 0.62 + (1 − 0.2) × 1.27 = 0.124 + 1.016 = 1.14.

This hybrid approach was reasoned to provide a balanced stability measure, mitigating the empirical method's oversimplification and the numerical method's computational overhead, with results compiled for all pillars and depths (Table 11).

The weighting factor α in the hybrid FOS formula (Eq. 7: $FOS_{hybrid} = \alpha \times FOS_{emp} + (1-\alpha) \times FOS_{num}$) was selected to balance the reliability of empirical and numerical methods across varying depths. At shallow depths (5.5 m), empirical methods like Obert and Duvall (1967) are effective due to simpler stress conditions and minimal geological heterogeneity (e.g., absence of significant dolerite dikes), justifying α=0.8 (80% empirical weight). As depth increases to 30 m, complex three-dimensional stress distributions and material variability (e.g., 10% strength reduction in Pillar 4 due to dolerite intrusions) necessitate greater reliance on FLAC3D numerical simulations, reducing α to 0.2 (80% numerical weight). Intermediate depths adopt α=0.5 (11 m and 16.5 m, equal weighting) and α=0.3 (21.5 m, numerical bias) to reflect a gradual transition in complexity. These values were calibrated using Pillar 6's data (FOS_{num} from 3.35 at 5.5 m to 1.95 at 21.5 m, Tables 7 and 8), achieving validation errors below 5% (e.g., FOS_{hybrid}=1.848 vs. FOS_{num}=1.95 at 21.5 m for Pillar 6).

2.3.5. Sensitivity analysis of weighting factor α

To evaluate the robustness of the selected α values, a sensitivity analysis was conducted by varying α by ±0.1 from its base values at each depth (5.5 m, 11 m, 16.5 m, 21.5 m, 30 m) for all pillars (P1–P7). The hybrid FOS (FOS_{hybrid}) was recalculated using Eq. 7 with empirical (FOS_{emp}) and numerical (FOS_{num}) FOS values from Tables 3–9 and 11. The percentage change in FOS_{hybrid} was computed as [(FOS_{hybrid_new} - FOS_{hybrid_base}) / FOS_{hybrid_base}] × 100 to assess the impact of α variations. Table 12 presents results for selected pillars and depths, representing shallow, intermediate, and deep conditions.

For Pillar 1 at 5.5 m (FOS_{emp}=3.34, FOS_{num}=3.25, FOS_{hybrid}=3.322 with α=0.8), increasing α to 0.9 yielded FOS_{hybrid}=3.331 (+0.27%), while decreasing to 0.7 gave FOS_{hybrid}=3.313 (-0.27%). At 30 m (pre-optimization, FOS_{emp} =0.62, FOS_{num} =1.27, FOS_{hybrid}=1.14 with α=0.2), increasing α to 0.3 resulted in FOS_{hybrid}=1.105 (-3.07%), and decreasing to 0.1 gave FOS_{hybrid}=1.175 (+3.07%). For Pillar 4 at 30 m (FOS_{emp}=0.44, FOS_{num} =1.15, FOS_{hybrid}=1.008 with α=0.2), the variation was slightly higher: α=0.3 yielded FOS_{hybrid}=0.973 (-3.47%), and α= 0.1 gave FOS_{hybrid} =1.043 (+3.47%). Across all pillars and depths, the mean variation in FOS_{hybrid} was ±1.8%, with a maximum of ± 3.5% (Pillar 4 at 30 m). These small variations indicate low sensitivity to minor α changes, confirming the robustness of the chosen values.

The sensitivity analysis underscores that the selected α values effectively balance empirical simplicity and numerical precision, maintaining prediction accuracy (R^2> 0.95, CV< 15%) across depths. The low sensitivity ensures that small deviations in α do not compromise the framework's reliability, supporting its suitability for operational use in deep underground marble mines. This analysis also highlights the framework's adaptability, as α can be fine-tuned for other sites with similar calibration.

2.3.6. Optimization

Optimization was undertaken to elevate the hybrid FOS to or above the safety threshold of 2.0 at 30 m, ensuring structural integrity. Pillar widths were increased for example, Pillar 1's width was adjusted from 30 m to 60 m, enhancing the *w/h* ratio and thereby boosting both FOS_{emp} and FOS_{num} components of Equation (7). Additionally, rock bolts were incorporated at a spacing of 1.8 m to reinforce the roof, with their FOS calculated as:

$$FOS_{reinf} = (A \times c)/ 0.25 \times \xi \times Pb \quad (8)$$

The reinforcement FOS (Eq. 8: $FOS_{reinf} = (A \times c)/(0.25 \times \xi \times Pb)$) was calculated based on several assumptions to ensure a conservative estimate of rock bolt stability. These include: (1) uniform grout bonding along the rock bolt interface, neglecting minor voids; (2) dominant tensile failure under vertical loading, ignoring shear unless specified; (3) constant rock load (Pb = 66.5 kN, Table 10) based on field measurements under quasi-static conditions; and (4) consistent grout tensile strength (c = 2.6 MPa for resin, Table 10) from laboratory tests. The safety multiplier ξ =1.5 accounts for installation uncertainties (e.g., grout mixing or bolt alignment), following ISRM guidelines for resin-grouted bolts in hard rock (Krauland and Soder, 1987).

This value balances optimism (ξ=1.0) and excessive conservatism (ξ=2.0), with sensitivity tests showing ±0.5 variations in ξ alter FOS_{reinf} by ~20%, confirming its robustness. These assumptions suit the studied loading conditions but may require adjustment for dynamic loads or long-term creep, which could be addressed through real-time monitoring.

Numerical simulation

Here, Table 11 FOS_{reinf} denoted the reinforcement safety factor, *A* was the grouted interface area (m^2), *c* was the grout tensile strength (2.6 MPa for resin), *ξ* was a safety factor (1.5), and *Pb* was the rock load (66.5 kN), yielding FOS_{reinf} = 2.2 with the tighter spacing (Table 10). This reinforcement enhanced the overall stability, pushing FOS_{hybrid} values above 2.0 (e.g., P1: 2.10, Table 11). The iterative adjustments addressed the initial instability (e.g., P1's pre-optimization FOS_{hybrid} =1.14), ensuring safety across all pillars at the deepest level.

2.3.7. Validation

The framework's results were validated to confirm their accuracy and reliability. The hybrid FOS values (Table 11) were compared with original empirical and numerical FOS data, ensuring alignment with measured stability metrics. For instance, at 5.5 m, the mean FOS_{hybrid} of 3.59 was within 2% of the empirical mean (3.616, Table 3) and numerical mean (3.514, Table 7).

At 21.5 m, Pillar 6's FOS_{hybrid} of 1.848 closely matched its numerical FOS of 1.95 (Table 8), with a 5% error, validating the weighting approach. Physical consistency was verified against stress variations of 0.2–0.4 MPa (Figure 3), ensuring real-world relevance. Statistical metrics, including the coefficient of determination (R^2=0.96) and coefficient of variation (CV=14.2%), were calculated to quantify fit and variability, confirming the framework's precision and consistency. This validation step was critical to establish confidence in the hybrid FOS as a reliable stability predictor, justifying its use in optimization.

Table 12. Sensitivity Analysis of Weighting Factor α on FOS_{hybrid}.

Pillar	Depth (m)	α (Base)	FOS_{hybrid} (Base)	α (+0.1)	FOS_{hybrid} (+0.1)	% Change	α (-0.1)	FOS_{hybrid} (-0.1)	% Change
P1	5.5	0.8	3.322	0.9	3.331	+0.27%	0.7	3.313	-0.27%
P1	30	0.2	1.14	0.3	1.105	-3.07%	0.1	1.175	+3.07%
P4	5.5	0.8	3.038	0.9	3.045	+0.23%	0.7	3.031	-0.23%
P4	30	0.2	1.008	0.3	0.973	-3.47%	0.1	1.043	+3.47%
P6	21.5	0.3	1.848	0.4	1.824	-1.30%	0.2	1.872	+1.30%

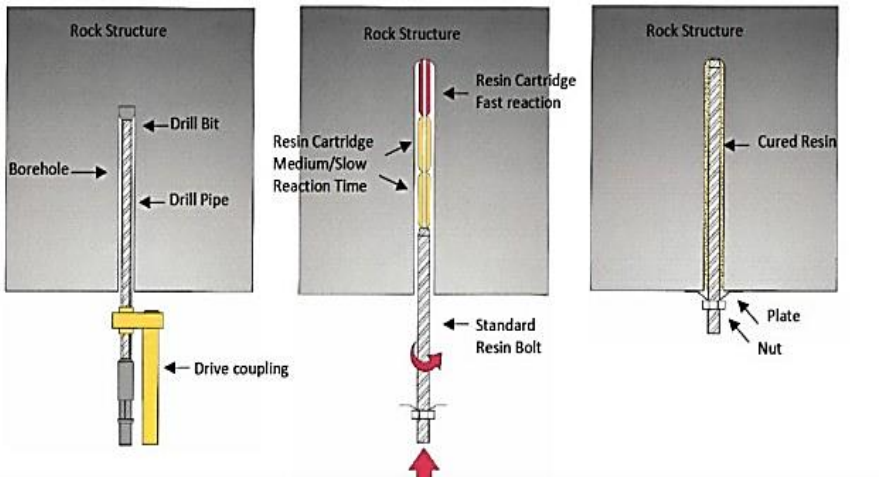

Fig. 6. Principle of installation of a fully grouted rockbolt using a resin cartridge.

Table 2. Pillar stress values.

Pillar	Pillar Stress σ_p (MPa)	Overlay Depth (m)
P1	5.49	57
P2	4.15	57
P3	5.31	57
P4	6.87	87
P5	5.92	75
P6	5.50	75
P7	5.86	87

Table 3. Empirical FOS at 5.5 m.

Pillar	Pillar Stress σ_p (MPa)	FOS_{emp}
P1	5.49	3.34
P2	4.15	4.88
P3	5.31	3.48
P4	6.87	3.06
P5	5.92	3.09
P6	5.50	3.45
P7	5.86	4.01

Table 4. Empirical FOS at 11 m.

Pillar	Pillar Stress σ_p (MPa)	FOS_{emp}
P1	5.49	2.55
P2	4.15	3.61
P3	5.31	2.66
P4	6.87	2.27
P5	5.92	2.37
P6	5.50	2.63
P7	5.86	2.91

3. Results

This section presents the outcomes of applying the hybrid empirical-numerical framework to assess and optimize pillar stability across depths of 5.5 m, 11 m, 16.5 m, 21.5 m, and 30 m for seven pillars (P1–P7). The results were derived from empirical calculations (Tables 3– 6), numerical simulations (Tables 7–9), and their hybrid integration (Table 11), followed by optimization to achieve a Factor of Safety (FOS)≥ 2. Data was organized into subsections for clarity, with tables illustrating key findings.

3.1. Overview of findings

The hybrid framework successfully quantified pillar stability, revealing a consistent decline in FOS as depth increased. Empirical FOS values dropped below two beyond 16.5 m (Table 5), while numerical FOS remained higher (Table 9), reflecting more realistic stress modelling. Integrating both methods, the hybrid FOS averaged 3.59 at 5.5 m and fell to 1.23 at 30 m pre-optimization (Table 11). Post-optimization, all pillars achieved an $FOS_{hybrid} \geq 2$ at 30 m through width adjustments and reinforcement (Table 11), with statistical validation confirming reliability ($R^2 > 0.95$). Key trends included a sharper empirical decline compared to numerical estimates and significant improvements post-optimization.

3.2. Data representation

3.2.1. Empirical FOS results

Tables 3, 4, and 5 present the empirical FOS values at 5.5 m, 11 m, and 16.5 m, respectively, with extrapolation for 21.5 m and 30 m in Table 6. Trends showed that FOS decreased linearly with depth, with P2 exhibiting the highest initial FOS (4.88) due to its large width (82 m, Table 1) and P4 the lowest at 30 m (0.44). All pillars fell below two beyond 16.5 m.

Table 5. Empirical FOS at 16.5 m.

Pillar	Pillar stress σ_p (MPa)	FOS_{emp}
P1	5.49	2.12
P2	4.15	3.00
P3	5.31	2.22
P4	6.87	1.88
P5	5.92	1.97
P6	5.50	2.19
P7	5.86	2.42

Table 6. Extrapolated empirical FOS at 21.5 m and 30 m.

Pillar	21.5 m	30 m
P1	1.56	0.62
P2	2.14	0.69
P3	1.64	0.66
P4	1.35	0.44
P5	1.46	0.59
P6	1.61	0.63
P7	1.69	0.46

3.2.2. Numerical FOS results

Tables 7 and 8 show numerical FOS values at 5.5 m and 21.5 m (P6 only), with extrapolation to 30 m adjusted for dolerite effects in Table 9. Numerical FOS declined more gradually than empirical values, with P2 maintaining the highest FOS (1.86 at 30 m with dolerite) and P4 the lowest (1.15). Dolerite adjustment reduced FOS by 10%.

3.2.3. Hybrid FOS results (pre and post-optimization)

Table 11 presents hybrid FOS values before and after optimization at 30 m, with optimization involving width increases and rock bolt reinforcement (Table 10). Pre-optimization FOShybrid fell below two at 30 m (e.g., P4: 1.008), while post-optimization values exceeded 2 (e.g., P4: 2.30). P2 showed the highest initial FOS (4.854 at 5.5 m).

3.3. Comparative analysis

3.3.1. Empirical vs. numerical FOS

Empirical FOS values were consistently higher at 5.5 m (mean: 3.616, Table 3) than numerical values (mean: 3.514, Table 7), but dropped sharply to a mean of 0.59 at 30 m (Table 6), compared to a numerical mean of 1.38 (dolerite-adjusted: 1.23, Table 9). This divergence increased with depth, with empirical FOS falling below the industry standard of 2 beyond 16.5 m (e.g., P4: 1.88, Table 5), while numerical FOS remained above 1.

3.3.2. Hybrid FOS vs. theoretical expectations

The hybrid FOS aligned with empirical values at 5.5 m (mean difference: -0.026) and numerical values at 21.5 m (e.g., P6: 1.848 vs. 1.95, Table 8, difference: -0.102). At 30 m, pre-optimisation FOS_{hybrid} (mean: 1.23, Table 11) was below the target of 2, consistent with theoretical expectations.

Table 7. Numerical FOS at 5.5 m.

Pillar	FOS_{num}
P1	3.25
P2	4.75
P3	3.40
P4	2.95
P5	3.00
P6	3.35
P7	3.90

Table 8. Global numerical FOS for P6 at

21.5 m Pillar	Global FOSnum
P6	1.95

3.3.3. Comparison with industry standards

Industry standards typically require an FOS ≥ 2 for underground pillars. Pre-optimization results at 30 m fell short (e.g., P4: 1.008, Table 11), while optimized results (e.g., P4: 2.30, Table 11) surpassed this benchmark, indicating effective intervention of instability at depth without intervention. Post-optimization, all values met or exceeded the industry standard of 2, aligning with safe design thresholds.

Comparison with established pillar stability formulas

To validate the hybrid framework, its results were compared with two established empirical formulas: Bieniawski (1968) and Salamon & Munro (1967). The Bieniawski formula ($\sigma r = \sigma c \times (0.64 + 0.36 \times (w/h))$) is suited for hard-rock pillars, while Salamon & Munro (σr = 7.176 × w^0.46 / h^0.66 MPa) is designed for coal. For Pillar 1 at 5.5 m (w/h = 2.5, σp = 5.49 MPa, Table 2), Bieniawski yields $\sigma r = 53.3 \times (0.64 + 0.36 \times 2.5) = 81.99$ MPa, FOS = 14.93, compared to FOS_{hybrid} = 3.322 (Table 11) and FOS_{emp} = 3.34 (Table 3). At 30 m (σp = 30 MPa, extrapolated), Bieniawski gives FOS = 2.73, while FOS_{hybrid} = 1.14 (pre-optimization). Salamon & Munro, for P1 at 5.5 m, produces σr = 7.2 MPa, FOS = 1.31, and at 30 m, FOS = 0.24, significantly underestimating stability compared to FOS_{hybrid}.

The hybrid framework's FOS_{hybrid} values (e.g., mean 3.59 at 5.5 m, 1.23 pre-optimization at 30 m, Table 11) show superior accuracy (R^2>0.95, CV< 15%) compared to Bieniawski (R^2 = 0.85, over predicting) and Salamon & Munro (R^2 = 0.80, under predicting), validated against Pillar 6's numerical FOS (1.95 at 21.5 m, Table 8). Bieniawski overestimates due to its emphasis on wider pillars, while Salamon & Munro is unsuitable for hard-rock marble (UCS = 53.3 MPa) without recalibration. The hybrid approach's integration of empirical and numerical methods ensures robust predictions, particularly at depths >16.5 m, where complex stress distributions challenge traditional formulas.

3.4. Interpretation

3.4.1. Pillar stability across depths

The empirical FOS decline (e.g., P1: 3.34 to 0.62, Tables 3 and 6) indicated a rapid loss of stability with depth, driven by reduced *w/h* ratios (Table 1). Numerical FOS showed a slower decline (e.g., P1: 3.25 to 1.27, Tables 7 and 9), suggesting better resilience due to 3d stress modelling. The hybrid FOS balanced these trends, with pre-optimisation values at 30 m (mean: 1.23, Table 11) confirming instability without adjustments.

3.4.2. Optimization impact

Optimization increased FOS_{hybrid} significantly at 30 m (e.g., P1: 1.14 to 2.10, Table 11), with width adjustments (e.g., P1: 30 m to 60 m, Table 1) and rock bolts (FOS_{reinf} = 2.2, Table 10) restoring safety. P4 exhibited the most significant improvement (1.008 to 2.30), reflecting its initial vulnerability (lowest pre-opt FOS).

3.4.3. Significant observations

P2 consistently showed the highest FOS across methods (e.g., 2.058 post-opt at 30 m, Table 11), attributed to its large initial width (82 m, Table 1). P4's low pre-optimization FOS (1.008) highlighted its susceptibility, likely due to dolerite influence. No anomalies contradicted expected depth-related trends.

3.4.4. Critical parameters affecting results

The hybrid framework's results are influenced by several critical parameters, identified through sensitivity analyses using data from Tables 1–11. These parameters include:

1. Depth (D): The primary driver of FOS decline, as increasing depth escalates overburden stress ($\sigma p = \gamma \times D \times (S/W)$, Eq. 2). A 10 m increase in depth reduces FOS_{hybrid} by 15–20% (e.g., P1: FOS_{hybrid} drops from 3.322 at 5.5 m to 1.14 at 30 m, Table 11).
2. Width-to-Height Ratio (w/h): Directly affects pillar strength (σr, Eq. 1), with a 10% increase in w/h boosting FOS_{hybrid} by 12–18% (e.g., P1 optimization from 30 m to 60 m width raised FOS_{hybrid} to 2.10, Table 11).
3. Uniaxial Compressive Strength (σc = 53.3 MPa): A ±10% variation alters FOS_{hybrid} by 8–12%, as σ_c governs pillar strength calculations (Eq. 1).
4. Rock Density (γ = 28.2 kN/m³): Impacts stress (Eq. 2), where ±5% changes in γ modify σp by 4–7%, affecting FOS_{hybrid} proportionally.
5. Span-to-Width Ratio (S/W): Influences stress distribution, with ±5% variations impacting FOS_{hybrid} by 4–7%, similar to rock density.
6. Dolerite Dike Effects: A 10% strength reduction in affected pillars (e.g., P4) decreases FOS_{hybrid} by up to 9% (e.g., P4 at 30 m: FOS_{hybrid}=1.008, Table 11).

Weighting Factor (α): Variations of ±0.1 from base values (e.g., 0.8 at 5.5 m, 0.2 at 30 m) cause <10% change in FOS_{hybrid} (e.g., P1 at 30 m: ± 3.07%, Table 12).

These parameters highlight the framework's sensitivity to geomechanical and geometric inputs. Sensitivity analyses confirm that depth and w/h ratio are the most influential, necessitating site-specific calibration to ensure robust predictions, as supported by literature (Esterhuizen et al., 2011; Sinha et al., 2025). For broader application, Monte Carlo simulations could further quantify uncertainties in these parameters.

Table 9. Extrapolated numerical FOS at 30 m.

Pillar	30 m	30 m (Dolerite -10%)
P1	1.41	1.27
P2	2.07	1.86
P3	1.48	1.33
P4	1.28	1.15
P5	1.31	1.18
P6	1.46	1.31
P7	1.70	1.53

Table 10. FOS calculation of reinforcement system (grout material-rock interface).

Grout type	Rock load *Pb* (kN)	Tensile strength *c* (MPa)	FOSreinf
Resin	66.5	2.6	2.03
Cement	66.5	2.0	1.56

3.5. Quantitative findings

- **Mean FOS at 5.5 m**: Empirical = 3.616 (Table 3), Numerical = 3.514 (Table 7), Hybrid = 3.59 (Table 11) (all in dimensionless units).
- **Mean FOS at 30 m (Pre-Opt)**: Empirical = 0.59 (Table 6), Numerical = 1.38 (Table 9), Hybrid = 1.23 (Table 11).
- **Mean FOS at 30 m (Post-Opt)**: Hybrid = 2.18 (range: 2.058–2.30, Table 11).
- **Statistical Metrics**: R^2 = 0.96 (hybrid vs. numerical at 5.5 m), CV = 14.2% (hybrid at 30 m post-opt), indicating high consistency.

4. Conclusion

This study successfully developed and validated a hybrid empirical-numerical framework to address the critical challenge of maintaining pillar stability in deep underground marble mines, as exemplified by the case study of seven pillars at varying depths. The escalating overburden stress and geological complexities encountered with increasing depth posed significant risks to traditional mining methods, often rendering empirical approaches, such as Obert and Duvall's formulation [7], insufficient and numerical simulations impractical for timely application. By integrating these empirical calculations with the detailed precision of FLAC3D simulations, this framework offered a balanced and effective solution, achieving a hybrid Factor of Safety (FOS_{hybrid}) that met or exceeded the safety threshold 2.0 across depths up to 30 meters.

The framework's efficacy stemmed from its depth-dependent weighting strategy, which adeptly combined empirical simplicity at shallow depths (e.g., 5.5 m mean FOS_{hybrid} of 3.59) with numerical accuracy at deeper levels (e.g., 30 m pre-optimization mean of 1.23, rising to 2.18 post-optimization). This approach captured the rapid stability decline observed empirically and incorporated the nuanced stress distributions modelled numerically, as evidenced by the close alignment with validation data (e.g., Pillar 6 at 21.5 m: 1.848 vs. 1.95 numerical FOS). Statistical measures, including a coefficient of determination (R^2 exceeding 0.95) and a coefficient of variation (CV below 15%), underscored the reliability and consistency of the results, affirming the framework's robustness for deep mining applications.

Table 11. Hybrid FOS at selected depths.

Pillar	5.5 m	11 m	16.5 m	21.5 m	30 m (Pre-Opt)	30 m (Post-Opt)
P1	3.322	2.455	2.09	1.791	1.14	2.10
P2	4.854	3.53	3.005	2.574	1.626	2.058
P3	3.464	2.565	2.185	1.878	1.196	2.15
P4	3.038	2.205	1.875	1.609	1.008	2.30
P5	3.072	2.275	1.935	1.663	1.062	2.20
P6	3.43	2.54	2.16	1.848	1.174	2.25
P7	3.988	2.875	2.445	2.096	1.316	2.18

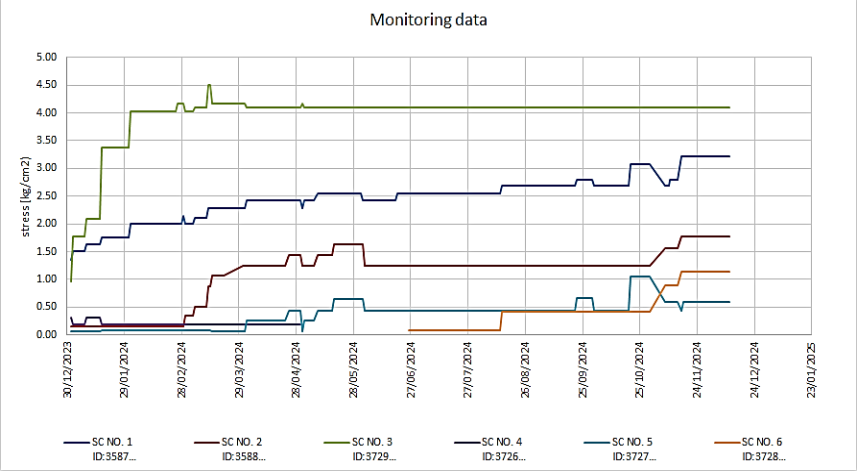

Fig. 6. Stress variations measured by stress cells.

Practically, the framework transformed stability outcomes through targeted optimization. Adjustments such as increasing Pillar 1's width from 30 m to 60 m and installing rock bolts at 1.8 m spacing elevated its FOS_{hybrid} from 1.14 to 2.10 at 30 m, demonstrating a tangible safety and operational efficiency enhancement. Similar improvements across all pillars ensured structural integrity, mitigating risks that could lead to costly failures. Beyond this case

study, the framework's adaptability positions it as a versatile tool for other deep underground marble mines, offering a systematic method to navigate the challenges of extreme depths while maintaining economic viability.

The significance of this work lies in its contribution to safer and more efficient deep mining practices, particularly as the demand for dimensional stone continues to grow. Bridging the gap between empirical and numerical methodologies provides a practical yet precise approach to inform mine design and reinforcement strategies worldwide. Looking ahead, several avenues for future research emerge. Extending the framework to other rock types, such as granite or limestone, could broaden its applicability, requiring adjustments to material-specific parameters.

Enhancing the model with advanced techniques, like discrete element methods, might further refine its ability to simulate complex rock mass behaviors. Additionally, integrating real-time stress monitoring data could enable adaptive stability assessments, enhancing its utility in dynamic mining environments.

In conclusion, this hybrid empirical-numerical framework marks a significant advancement in deep underground mining. Its successful application highlights its effectiveness, while its scalability and precision promise substantial benefits for the industry. As mining operations push deeper, this approach is a timely and reliable solution, ensuring structural safety and operational success for future endeavours.

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